Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 437–445
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

The pressing of an elastoviscoplastic material between rigid coaxial cylindrical surfaces☆

A.A. Burenin, L.V. Kovtanyuk, A.L. Mazelis

Vladivostok

Received 19 April 2005

Abstract

An analytical solution of the problem of the pressing of an incompressible elastoviscoplastic material between two rigid coaxial cylindrical surfaces is obtained using the theory of large strains. The velocity of motion of the elastic core, the change in the dimensions of the zones of irreversible deformation and the stresses and strains both in the elastic core and in the flow region are calculated as a function of the variable pressing force.

Exact solutions of the rectilinear motion of a viscoplastic medium,[1,2] which has already become classical, have served as a reliable basis for testing algorithms and programs for numerical calculations in non-linear boundary-value problems with unknown moving boundaries (the boundaries of the stagnation zones and the surfaces of rigid cores). In cases when it is not possible to neglect the elastic properties of the material (the calculation of the residual stresses, the prediction of the geometry of the constructions after manufacture, etc.), the analytical solution obtained below for an elastoviscoplastic material can serve the purpose of testing the appropriate calculation procedures.

1. Initial model relations

The basic hypothesis for constructing a model of large strains is the separation of the observed total strains into experimentally non-measurable reversible and irreversible components. It was assumed in Ref. 3,4 that the irreversible strains can only be stored in a material under conditions in which the stress states correspond to loading surfaces in stress space, while during unloading or irreversible strain the components of the inverse strain tensor vary as if the body had rigidly displaced. This assumption enables us to specify more exactly the differential equations of the change (transfer Ref. 5) of the tensors of the component strains: the reversible strain e_{ij} and the irreversible strain p_{ij}, which must be assumed by the definitions of these tensors. In a Cartesian system of coordinates, these equations can be written as follows[3]:

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 3, pp. 481–489, 2006.
E-mail address: burenin@iacp.dvo.ru (A.A. Burenin).

0021-8928/$ – see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2006.07.004

$$\frac{de_{ij}}{dt} = \varepsilon_{ij} - \varepsilon_{ij}^p - \frac{1}{2}(e_{ik}\upsilon_{k,j} + \upsilon_{k,i}e_{kj} - r_{ik}e_{kj} + e_{ik}r_{kj} - \varepsilon_{ik}^p e_{kj} + e_{ik}\varepsilon_{kj}^p)$$
$$\frac{dp_{ij}}{dt} = \varepsilon_{ij}^p - p_{is}r_{sj} - p_{is}\varepsilon_{sj}^p + r_{is}p_{sj} - \varepsilon_{is}^p p_{sj}$$
$$\upsilon_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \upsilon_j u_{i,j}, \quad u_{i,j} = \frac{\partial u_i}{\partial x_j} \tag{1.1}$$
$$\varepsilon_{ij} = \frac{1}{2}(\upsilon_{i,j} + \upsilon_{j,i}), \quad r_{ij} = \frac{1}{2}(\upsilon_{i,j} - \upsilon_{j,i}) + F_{ij}(\varepsilon_{st}, e_{st})$$

Here u_i and υ_i are the components of the displacement and velocity vectors of points of the medium, ε_{ij} will be called the components of the tensor of the rates of plastic deformations, in accordance with the classical theory of plasticity, and r_{ij} are the components of the rotation tensor, which depend, in its non-linear part, on the reversible strains and rates of deformation. There is an explicit expression for F_{ij} in Ref. 3. According to Eqs. (1.1) the components of the total Almansi strains can be represented in terms of the components of the constituent tensors e_{ij} and p_{ij} in the form

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}) = e_{ij} + p_{ij} - \frac{1}{2}e_{ik}e_{kj} - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{ks}e_{sj} \tag{1.2}$$

Using Eqs. (1.1), as a consequence of the first law of thermodynamics we can obtain the following analogues of Murnahan's formula

$$\sigma_{ij} = -P\delta_{ij} + \frac{\partial W}{\partial d_{ik}}(\delta_{kj} - 2d_{kj}) \quad \text{for} \quad p_{ij} \equiv 0$$
$$\sigma_{ij} = -P_1\delta_{ij} + \frac{\partial W}{\partial e_{ik}}(\delta_{kj} - e_{kj}) \quad \text{for} \quad p_{ij} \neq 0 \tag{1.3}$$

Here we have taken into account the condition of incompressibility of the medium, and P and P_1 are additional hydrostatic pressures. Henceforth we will assume that the medium is isotropic and we will taken the following relation for the elastic potential of the medium

$$W = W(\Lambda_1, \Lambda_2) = -2\mu\Lambda_1 - \mu\Lambda_2 + b\Lambda_1^2 + (b - \mu)\Lambda_1\Lambda_2 - \chi\Lambda_1^3 + \cdots$$
$$\Lambda_m = \begin{cases} L_m & \text{for} \quad p_{ij} \equiv 0 \\ I_m & \text{for} \quad p_{ij} \neq 0 \end{cases} \tag{1.4}$$
$$L_1 = d_{kk}, \quad L_2 = d_{ik}d_{ki}; \quad I_1 = e_{kk} - \frac{1}{2}e_{sk}e_{ks}, \quad I_2 = e_{st}e_{ts} - e_{sk}e_{kt}e_{ts} + \frac{1}{4}e_{sk}e_{kt}e_{tn}e_{ns}$$

where μ, b and χ are constants of the material, L_1 and L_2 are invariants of the Almansi tensor, and I_1 and I_2 are invariants of the reversible strain tensor. This choice of invariance enables us to take the limit from the second relation in (1.3) to the first relation as the irreversible strains approach zero.

We will define the dissipative mechanism of the deformation by specifying the load surface in the form

$$\max|\sigma_i - \sigma_j| = 2k + 2\eta\max|\varepsilon_k^p| \tag{1.5}$$

Here k is the yield point, η is the viscosity coefficient, σ_i are the components of the principal stresses and ε_k^p are the components of the principal rates of plastic deformation.

As in the classical theory of plasticity, we will assume the conditions of the Mises maximum principle. The load surface (1.5) is then the plastic potential, and the relation between the rates of irreversible strains and the stresses is established by the associated law of plastic flow

$$\varepsilon_{ij}^p = \lambda\frac{\partial f}{\partial \sigma_{ij}}, \quad f(\sigma_{ij}, \varepsilon_{ij}^p) = k, \quad \lambda > 0 \tag{1.6}$$

The plasticity condition (1.5) is an extension of the Tresca plasticity condition to the case when the viscous properties of the material in plastic flow are taken into account, and it can be considered as the fundamental relation of the model of an isotropically hardened plastic solid.[6,7]

2. The initial elastic equilibrium

Suppose an incompressible elastoviscoplastic material, the deformation properties of which are described above, forms a plug in the space between two rigid coaxial cylindrical surfaces $r = r_0$ and $r = R$ ($r_0 < R$) of height l. We will investigate the conditions under which, by creating a pressure on one side of the plug, it can be moved along the cylindrical surfaces as a result of plastic flow of the material of the plug in the surroundings of these surfaces. The solution of this boundary-value problem of the theory of large elastoviscoplastic deformations in a cylindrical system of coordinates, r, θ, z will be sought in the class of functions $u = u_z(r, t)$, $v = v_z(r, t)$ and $P = P(r, z, t)$. Hence, the boundary conditions of the problem will be

$$u(R,t) = u(r_0,t) = v(R,t) = v(r_0,t) = 0$$
$$\sigma_{zz}(r_*, u(r_*,t), t) = -p(t), \quad \sigma_{zz}(r_*, l + u(r_*,t), t) = 0 \tag{2.1}$$

The kinematic constraints in relations (2.1) are the conditions for the material to adhere to the rigid walls and $p(t)$ is the pressure produced on one side of the plug ($z = 0$). The coordinate $r = r_*$ is related to the maximum displacement u of the boundary points of the plug and will be calculated below.

We will assume that up to the instant of time $t_0 = 0$ ($p(0) = p_0$) the material has been deformed elastically, and plastic flow begins from this instant of time in the neighbourhood of the inner rigid wall. Hence, the elastic equilibrium state at the instant of time t_0 defines the initial condition for the subsequent process of irreversible deformation. We will calculate the parameters of this stress-strain state.

According to Eq. (1.2), the strain tensor has two non-zero components

$$d_{rr} = -\frac{1}{2}u_{,r}^2, \quad d_{rz} = \frac{1}{2}u_{,r} \tag{2.2}$$

From relations (1.3), (1.4) and (2.2) we obtain the following relations for the components of the stress tensor, apart from terms of the second order of smallness in the components of the displacement gradient,

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} - \mu u_{,r}^2 = -(P + 2\mu) - \frac{1}{2}(b + \mu)u_{,r}^2 = -s, \quad \sigma_{rz} = \mu u_{,r} \tag{2.3}$$

Using the equilibrium equations

$$\sigma_{rz,r} + \sigma_{zz,z} + \frac{\sigma_{rz}}{r} = 0, \quad \sigma_{rr,r} + \sigma_{rz,z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{2.4}$$

and boundary conditions (2.1), we obtain

$$u = \frac{R^2c}{4\mu}(\tilde{r}^2 - 1 - 2\tilde{r}_*^2\ln\tilde{r}), \quad s = c(z - l) - \frac{R^2c^2}{4\mu}(\tilde{r}_*^2(1 - 2\ln\tilde{r}_*) - 1)$$
$$p_0 = -cl, \quad \tilde{r}_* = \frac{r_*}{R} = \sqrt{\frac{\tilde{r}_0^2 - 1}{2\ln\tilde{r}_0}}, \quad \tilde{r}_0 = \frac{r_0}{R}, \quad \tilde{r} = \frac{r}{R} \tag{2.5}$$

To calculate the constant c, we will use the plasticity condition (1.5), which, in the case considered, takes the form

$$\sigma_{rz}|_{r = r_0} = k \tag{2.6}$$

The final relations for the stress components have the form

$$\sigma_{zz} = c(l-z) + \frac{R^2c^2}{4\mu}\left(\tilde{r}^2 - 1 + \frac{\tilde{r}_*^4}{\tilde{r}^2} - \tilde{r}_*^2(1 + 2\ln\tilde{r}_*)\right)$$
$$\sigma_{rr} = \sigma_{\theta\theta} = -s, \quad \sigma_{rz} = \frac{Rc}{2}\left(\tilde{r} - \frac{\tilde{r}_*^2}{\tilde{r}}\right), \quad c = \frac{2kR^{-1}}{\tilde{r}_0^2 - \tilde{r}_*^2} \tag{2.7}$$

Since only the stress state of the material will satisfy relations (2.7), a region of plastic flow begins to develop from the boundary $r = r_0$.

3. Deformation in the case of unilateral plastic flow

When the loading pressure is increased further $p(t) > p_0$ the region of irreversible deformation turns out to be bounded by the surfaces $r_0 \le r \le r_1(t)$. In the region $r_1(t) < r \le R$ the material remains in the elastic state. This situation will be maintained up to the instant of time t_1, at which a new region of irreversible deformation appears and begins to develop from the boundary surface $r = R$. We will indicate the parameters of the stress-strain state of the material in this time interval.

Remaining within the framework of the quasi-static approach (neglecting forces of inertia), from the equilibrium equations (2.4) we obtain for the region of reversible deformation

$$s(t) = c(t)z + a(t), \quad \sigma_{rz} = \frac{c(t)r}{2} + \frac{c_1(t)}{r}$$
$$u = \frac{R^2c(t)}{4\mu}(\tilde{r}^2 - 1) + \frac{c_1(t)}{\mu}\ln\tilde{r}, \quad v = \frac{R^2\dot{c}(t)}{4\mu}(\tilde{r}^2 - 1) + \frac{\dot{c}_1(t)}{\mu}\ln\tilde{r} \tag{3.1}$$

Here $a(t)$, $c(t)$ and $c_1(t)$ are unknown functions, where the dots indicate their derivatives with respect to time. When integrating the equilibrium equations we took into account the condition $u(R, t) = 0$.

Following Murnahan's formula (1.3) when $p_{ij} \neq 0$, we can obtain, for the stresses in the of plastic flow region,

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} - 4\mu e_{rz}^2 = -(P_1 + 2\mu) - 2(\mu + b)e_{rz}^2 = -s_1(t), \quad \sigma_{rz} = 2\mu e_{rz} \tag{3.2}$$

where we have taken into account the kinematic relations which hold in the case considered

$$e_{rr} = -\frac{3}{2}e_{rz}^2, \quad e_{zz} = \frac{1}{2}e_{rz}^2$$

On the other hand, by integrating the equilibrium equations in the plastic flow region $r_0 \le r \le r_1(t)$, we can obtain

$$s_1(t) = m(t)z + n(t), \quad \sigma_{rz} = \frac{m(t)r}{2} + \frac{g(t)}{r}, \quad e_{rz} = \frac{m(t)r}{4\mu} + \frac{g(t)}{4\mu r} \tag{3.3}$$

The conditions of continuity of the stresses at the elastoplastic boundary $r = r_1(t)$ make it necessary to assume

$$m(t) = c(t), \quad n(t) = a(t), \quad g(t) = c_1(t), \quad s(t) = s_1(t)$$

The plastic potential (1.5) in the region of irreversible deformation can be written in the form

$$f(\sigma_{rz}, \varepsilon_{rz}^p) = \sigma_{rz}^2 - (k + \eta\varepsilon_{rz}^p)^2 = 0 \tag{3.4}$$

Following the associated law of plastic flow (1.6), we obtain from relation (3.4)

$$\sigma_{rz} = k + \eta\varepsilon_{rz}^p, \quad \lambda = \frac{\varepsilon_{rz}^p}{k + \eta\varepsilon_{rz}^p} \tag{3.5}$$

Comparing formulae (3.3) and (3.5) we obtain

$$\varepsilon_{rz}^{p} = \frac{1}{\eta}\left(\frac{c(t)r}{2} + \frac{c_1(t)}{r} - k\right) \tag{3.6}$$

Substituting expressions (3.5) and (3.6) into relations (1.1) and (1.2), we can calculate the velocities of points and their displacements in the plastic flow region

$$\begin{aligned} v &= \frac{1}{\eta}h(t, r, r_0) - \frac{2k}{\eta}(r - r_0) + \frac{1}{2\mu}\dot{h}(t, r, r_0), \quad u = \int v(t)dt \\ h(t, r, r_0) &= \frac{c(t)}{2}(r^2 - r_0^2) + 2c_1(t)\ln\frac{r}{r_0} \end{aligned} \tag{3.7}$$

The conditions for the displacement and its derivatives to be equal on the elastoplastic boundary $r = r_1(t)$ lead to the relations

$$\begin{aligned} &c_2(t)r_1 + \frac{2c_3(t)}{r_1} - 2kt = 0 \\ &\frac{1}{2\mu}h(t, r_0, R) = \frac{1}{\eta}h_1(t, r_1, r_0) - \frac{2kt}{\eta}(r_1 - r_0), \quad \frac{1}{2\mu}\dot{h}(t, r_0, R) = \frac{1}{\eta}h(t, r_1, r_0) - \frac{2k}{\eta}(r_1 - r_0) \\ &c_2(t) = \int c(t)dt, \quad c_3(t) = \int c_1(t)dt; \quad h_1(t, r, r_0) = \frac{c_2(t)}{2}(r^2 - r_0^2) + 2c_3(t)\ln\frac{r}{r_0} \end{aligned} \tag{3.8}$$

The function $c_2(t)$, like $c(t)$, is given by the loading conditions (2.1): $c(t) = -l^{-1}p(t)$. From relations (3.8) we obtain for the unknown functions $c_1(t)$ and $c_3(t)$

$$\begin{aligned} c_3(t) &= ktr_1 - \frac{1}{2}c_2(t)r_1^2 \\ c_1(t) &= \frac{\mu}{\ln\tilde{r}_0}\left(\frac{R^2c(t)}{4\mu}(1 - \tilde{r}_0^2) + \frac{c_2(t)}{2\eta}\left(r_1^2 - r_0^2 - 2r_1^2\ln\frac{r_1}{r_0}\right) + \frac{2kt}{\eta}\left(r_0 - r_1 + r_1\ln\frac{r_1}{r_0}\right)\right) \end{aligned} \tag{3.9}$$

From relations (3.8) and (3.9) we obtain a differential equation which defines the position of the boundary of the plastic flow region $r = r_1(t)$

$$\dot{r}_1 = \frac{kr_1 - c_1(t) - c(t)r_1^2/2}{r_1c_2(t) - kt} \tag{3.10}$$

The stresses, both in the elastic and viscoplastic flow regions, are found in the usual way for the kinematic motion obtained. However, the solution obtained in this way only holds up to the instant of time t_1, when, in the neighbourhood of the external boundary surface $r = R$, the condition of plastic flow $\sigma_{rz}(R, t_1) = -k$ is satisfied, which, in the notation employed, can be written in the form

$$kR + \frac{1}{2}c(t_1)R^2 = -c_1(t_1) \tag{3.11}$$

Relation (3.11) is essentially the equation which defines the instant of time t_1 for the specified loading.

4. Calculation of the pressing process

Beginning from the instant t_1, two viscoplastic flow regions, bounded by the surfaces $r_0 \le r \le r_1(t)$ and $r_2(t) \le r \le R$, are present in the deformed material. In the region $r_1(t) < r < r_2(t)$ the pressed material remains in the elastic state. The parameters of the stress-strain state in this region are calculated in exactly the same way as the previous relations (3.1)

$$s(t) = b(t)z + a_1(t), \quad \sigma_{rz} = s_2(t,r), \quad s_2(t,r) = \frac{b(t)r}{2} + \frac{b_1(t)}{r}$$
$$u = \frac{b(t)r^2}{4\mu} + \frac{b_1(t)}{\mu}\ln r + d(t), \quad \upsilon = \dot{u} \tag{4.1}$$

In the viscoplastic flow region $r_0 \le r \le r_1(t)$, the displacements and the velocities of the points are calculated using the same relations (3.7), in which the functions of time $c(t)$, $c_1(t)$, $c_2(t)$ and $c_3(t)$ must be replaced by their next values $b(t)$, $b_1(t)$, $b_2(t)$ and $b_3(t)$.

When writing the plastic potential in the region $r_2(t) \le r \le R$ one must take into account that $\sigma_{rz} < 0$ and $\varepsilon_{rz}^p < 0$. Then, in the case considered it follows from relations (1.5) and (1.6) that

$$\sigma_{rz} = -k + \eta\varepsilon_{rz}^p, \quad \lambda = \frac{\varepsilon_{rz}^p}{\eta\varepsilon_{rz}^p - k} \tag{4.2}$$

The kinematics of the viscoplastic flow in the region considered is found by the same method by which relations (3.7) were previously obtained. This will have the same form as (3.7) with the functions $b(t)$, $b_1(t)$, $b_2(t)$ and $b_3(t)$, in which, however, r_0 must be replaced by R, and the second term in the first relation is taken with a plus sign. Finally, when calculating the displacement (a function of r and t) as the integral of the velocity with respect to time, unlike (3.7) the function of r, which appears in the integration, will now be non-zero and will be equal to

$$f(r) = -\frac{1}{\eta}h_1(t_1, r, R) - \frac{2kt_1}{\eta}(r - R)$$

The conditions of continuity of the displacements, velocities and the derivatives of the displacements with respect to r on the elastoplastic boundaries $r = r_1(t)$ and $r = r_2(t)$ serve to determine the unknown functions $b_3(t)$, $b(t)$ and $b_1(t)$, and also enable us to write the equations of motion of the boundaries: a differential equation for $r = r_2(t)$ and an algebraic equation for $r = r_1(t)$

$$b_3(t) = ktr_1 - \frac{b_2(t)r_1^2}{2}$$
$$d(t) = \frac{b_2(t)}{2\eta}\left(r_1^2 - r_0^2 - 2r_1^2\ln\frac{r_1}{r_0}\right) - \frac{2kt}{\eta}\left(r_1 - r_0 - r_1\ln\frac{r_1}{r_0}\right) - \frac{b(t)r_0^2}{4\mu} - \frac{b_1(t)}{\mu}\ln r_0$$
$$b_1(t) = \frac{\mu}{\ln\tilde{r}_0}\left(\frac{b_2(t)}{2\eta}\left(r_1^2 - r_0^2 - r_2^2 + R^2 - 2r_1^2\ln\frac{r_1R}{r_2r_0}\right) + \right.$$
$$\left. + \frac{2kt}{\eta}\left(r_0 - r_1 + R - r_2 + r_1\ln\frac{r_1R}{r_2r_0}\right) + \frac{R^2b(t)}{4\mu}(1 - \tilde{r}_0^2) - f(r_2)\right) \tag{4.3}$$
$$\dot{r}_2 = -\frac{kr_2 + b(t)r_2^2/2 + b_1(t)}{(b_2(t) - c_2(t_1))r_2 + k(t - t_1)}$$
$$r_1 = \frac{kt}{b_2(t)} + \sqrt{\left(\frac{kt}{b_2(t)}\right)^2 - \frac{(c_2(t_1) - b_2(t))r_2^2 + 2k(t_1 - t)r_2 + 2c_3(t_1)}{b_2(t)}}$$

The functions $a_1(t)$, $b(t) = -l^{-1}p(t)$ and $b_2(t) = \int b(t)dt$ are determined by the loading conditions.

Hence, the final solution of the boundary-value problem of the theory of elastovisco-plastic deformation in question is given by the following relations:

in the elastic core region $r_1(t) < r < r_2(t)$

$$u = \frac{1}{2\mu}H(t, r, r_0) + \frac{b_2(t)}{2\eta}\left(r_1^2 - r_0^2 - 2r_1^2\ln\frac{r_1}{r_0}\right) - \frac{2kt}{\eta}\left(r_1 - r_0 - r_1\ln\frac{r_1}{r_0}\right)$$
$$\upsilon = \frac{1}{2\mu}\dot{H}(t, r, R) + \frac{1}{\eta}H(t, r_2, R) - \frac{2k}{\eta}(R - r_2) \qquad (4.4)$$

in the viscoplastic flow regions

$$r_0 \le r \le r_1(t):\ u = \frac{1}{\eta}\int H(t, r, r_0)dt + \frac{1}{2\mu}H(t, r, r_0) - \frac{2kt}{\eta}(r - r_0), \quad \upsilon = \dot{u} \qquad (4.5)$$

$$r_2(t) \le r \le R:\ u = \frac{1}{\eta}\int H(t, r, R)dt + \frac{1}{2\mu}H(t, r, R) + \frac{2kt}{\eta}(r - R) + f(r), \quad \upsilon = \dot{u} \qquad (4.6)$$

The stresses in all three regions are given by the relations

$$\sigma_{rr} = \sigma_{\theta\theta} = H_1(t, z), \quad \sigma_{zz} = H_1(t, z) + \frac{1}{\mu}s_2^2(t, r), \quad \sigma_{rz} = s_2(t, r) \qquad (4.7)$$

In relations (4.5)–(4.7)

$$H(t, r, r_0) = \frac{b(t)}{2}(r^2 - r_0^2) + 2b_1(t)\ln\frac{r}{r_0}$$

$$H_1(t, z) = b(t)\left(l - z + \frac{1}{2\mu}H(t, r_*, r_0) + \frac{b_2(t)}{2\eta}\left(r_1^2 - r_0^2 - 2r_1^2\ln\frac{r_1}{r_0}\right) - \right.$$
$$\left. - \frac{2kt}{\eta}\left(r_1 - r_0 - r_1\ln\frac{r_1}{r_0}\right)\right), \quad r_* = \sqrt{-\frac{2b_1(t)}{b(t)}}$$

The position of the elastoplastic boundaries is given by the solution of the equations from (4.3).

The case when, beginning at a certain instant of time t_2, the loading pressure becomes constant, namely,

$$p(t)|_{t \ge t_2} = p(t_2) = p_1 = \text{const}$$

is of interest.

The presence of a singularity of the function $p(t)$ at the instant of time $t = t_2$ leads to the following changes in relations (4.3)–(4.7). In relations (4.3) and (4.7) and in the first relation of (4.4), $b_2(t)$ must be replaced by the function

$$b_2(t_2) - b(t_2)(t_2 - t)$$

In the region $r_0 \le r \le r_1(t)$, it is necessary to add the following term to the relations for the displacements (4.5)

$$G(r_0) = \frac{1}{2\eta}(b_2(t_2) - b(t_2)t_2)(r^2 - r_0^2)$$

and the term $G(R)$ to the similar relation (4.6).

We will present some characteristic results of the calculations obtained for the following values of the constants

$$\frac{k}{\mu} = 0.00621, \quad \frac{p_0}{\mu} = 0.01408, \quad \frac{r_0}{l} = 0.2, \quad \frac{R}{l} = 0.8, \quad \frac{\eta\alpha}{\mu} = 0.004$$

where the shear modulus $\mu = 8.05 \times 10^{10}$ Pa.

In Fig. 1 we show the development of the zones of viscoplastic flow with time for a linear loading function $p(t) = p_0(1 + \alpha t)$, where $\alpha > 0$ and p_0 corresponds to the occurrence of flow at the surface $r = r_0$. Note that, unlike a viscoplastic medium, flow at the rigid surfaces does not begin to develop simultaneously. The dependences of the dimensionless velocity $\upsilon\eta/(\mu l)$ and the displacement u/l on the dimensionless radius r/l are shown in Figs. 2 and 3

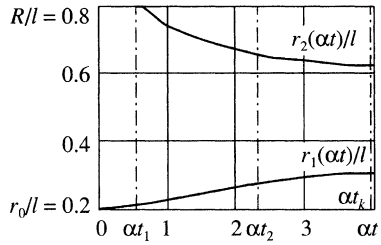

Fig. 1.

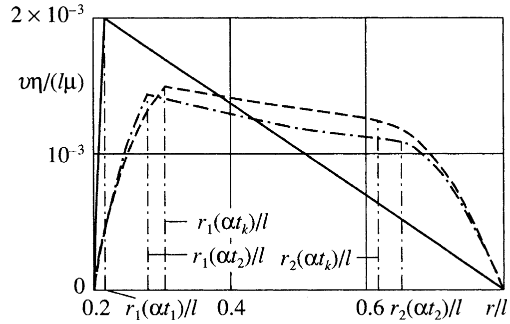

Fig. 2.

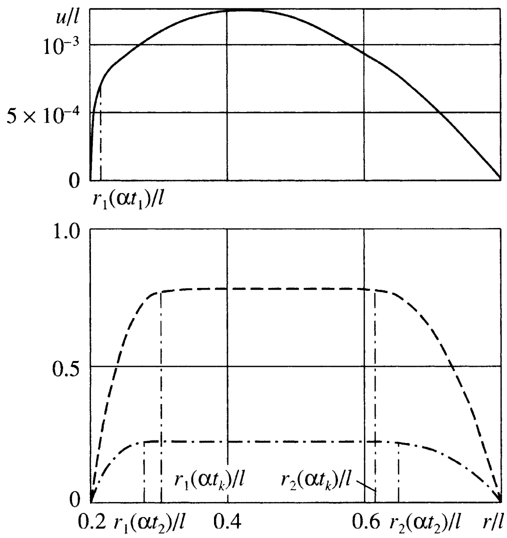

Fig. 3.

respectively at the instant of time αt_1, corresponding to the occurrence of flow on the surface $r = R$ (the continuous curves), and the instant of time αt_2 when the loading pressure becomes constant (the dash-dot curves) and at a certain current instant of time $\alpha t_k > \alpha t_2$ (the dashed curves). The variability of the velocity in the region of the solid core, unlike viscous plastic flow with a rigid core, is due to its being extended by elastic deformation.

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (05-01-00537-a), the State Programme for the Support of the Leading Scientific Schools (NSh 890.2003.1) and the State Programme for the Support of Young Russian Scientists – Candidates of Sciences and their Scientific Supervisors (MK-1774.2005.1).

References

1. Myasnikov VP. Some exact solutions for the rectilinear motions of a viscoplastic medium. *Zh Prikl Mekh Tekh Fiz* 1961;**2**:54–60.
2. Bykovtsev GI, Chernyshov AD. Viscoplastic flow in non-circular cylinders when there is a pressure drop. *Zh Prikl Mekh Tekh Fiz* 1964;**4**:94–6.
3. Burenin AA, Bykovtsev GI, Kovtanyuk LV. A simple model for an elastoplastic medium for finite deformations. *Dokl Ross Akad Nauk* 1996;**347**(2):199–201.
4. Kovtanyuk LV, Polonik MV. The Lamé problem of a thin-walled tube made of an incompressible elastoplastic material. In: *Problems of Continuum Mechanics and Structural Components*. Vladivostok: Dal'nauka; 1998.
5. Myasnikov VP. The equations of motion of elastoplastic materials for large strains. *Vestnik DVO Ross Akad Nauk* 1996;**4**:8–13.
6. Bykovtsev GI, Semykina TD. Viscoplastic flow of circular plates and shells of revolution. *Izv Akad Nauk SSSR Mekh Mashinostroyeniye* 1964;**4**:68–76.
7. Znamenskii VA, Ivlev DD. The equations of a viscoplastic body for piecewise-linear potentials. *Izv Akad Nauk SSSR Mekh Mashinostroyeniye* 1963;**6**:12–6.

Translated by R.C.G.